2. The Quantum String

Our goal in this section is to quantize the string. We have seen that the string action involves a gauge symmetry and whenever we wish to quantize a gauge theory we're presented with a number of different ways in which we can proceed. If we're working in the canonical formalism, this usually boils down to one of two choices:

- We could first quantize the system and then subsequently impose the constraints that arise from gauge fixing as operator equations on the physical states of the system. For example, in QED this is the Gupta-Bleuler method of quantization that we use in Lorentz gauge. In string theory it consists of treating all fields X^μ, including time X^0, as operators and imposing the constraint equations (1.33) on the states. This is usually called covariant quantization.

- The alternative method is to first solve all of the constraints of the system to determine the space of physically distinct classical solutions. We then quantize these physical solutions. For example, in QED, this is the way we proceed in Coulomb gauge. Later in this chapter, we will see a simple way to solve the constraints of the free string.

Of course, if we do everything correctly, the two methods should agree. Usually, each presents a slightly different challenge and offers a different viewpoint.

In these lectures, we'll take a brief look at the first method of covariant quantization. However, at the slightest sign of difficulties, we'll bail! It will be useful enough to see where the problems lie. We'll then push forward with the second method described above which is known as lightcone quantization in string theory. Although we'll succeed in pushing quantization through to the end, our derivations will be a little cheap and unsatisfactory in places. In Section 5 we'll return to all these issues, armed with more sophisticated techniques from conformal field theory.

2.1 A Lightning Look at Covariant Quantization

We wish to quantize D free scalar fields X^μ whose dynamics is governed by the action (1.30). We subsequently wish to impose the constraints

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0 \; . \tag{2.1}$$

The first step is easy. We promote X^μ and their conjugate momenta $\Pi_\mu = (1/2\pi\alpha')\dot{X}_\mu$ to operator valued fields obeying the canonical equal-time commutation relations,

$$\begin{aligned} [X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] &= i\delta(\sigma-\sigma')\,\delta^\mu{}_\nu \qquad , \\ [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] &= [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0 \quad . \end{aligned}$$

We translate these into commutation relations for the Fourier modes x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$. Using the mode expansion (1.36) we find

$$[x^\mu, p_\nu] = i\delta^\mu{}_\nu \quad \text{and} \quad [\alpha_n^\mu, \alpha_m^\nu] = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\,\eta^{\mu\nu}\delta_{n+m,0}\ , \tag{2.2}$$

with all others zero. The commutation relations for x^μ and p^μ are expected for operators governing the position and momentum of the center of mass of the string. The commutation relations of α_n^μ and $\tilde{\alpha}_n^\mu$ are those of harmonic oscillator creation and annihilation operators in disguise. And the disguise isn't that good. We just need to define (ignoring the μ index for now)

$$a_n = \frac{\alpha_n}{\sqrt{n}} \quad , \quad a_n^\dagger = \frac{\alpha_{-n}}{\sqrt{n}} \qquad\qquad n > 0 \tag{2.3}$$

Then (2.2) gives the familiar $[a_n, a_m^\dagger] = \delta_{mn}$. So each scalar field gives rise to two infinite towers of creation and annihilation operators, with α_n acting as a rescaled annihilation operator for $n > 0$ and as a creation operator for $n < 0$. There are two towers because we have right-moving modes α_n and left-moving modes $\tilde{\alpha}_n$.

With these commutation relations in hand we can now start building the Fock space of our theory. We introduce a vacuum state of the string $|0\rangle$, defined to obey

$$\alpha_n^\mu\,|0\rangle = \tilde{\alpha}_n^\mu\,|0\rangle = 0 \qquad\qquad \text{for} \;\; n > 0 \tag{2.4}$$

The vacuum state of string theory has a different interpretation from the analogous object in field theory. This is not the vacuum state of spacetime. It is instead the vacuum state of a single string. This is reflected in the fact that the operators x^μ and p^μ give extra structure to the vacuum. The true ground state of the string is $|0\rangle$, tensored with a spatial wavefunction $\Psi(x)$. Alternatively, if we work in momentum space, the vacuum carries another quantum number, p^μ, which is the eigenvalue of the momentum operator. We should therefore write the vacuum as $|0;p\rangle$, which still obeys (2.4), but now also

$$\hat{p}^\mu\,|0;p\rangle = p^\mu|0;p\rangle \tag{2.5}$$

where (for the only time in these lecture notes) we've put a hat on the momentum operator $\hat{p}^\mu$ on the left-hand side of this equation to distinguish it from the eigenvalue p^μ on the right-hand side.

We can now start to build up the Fock space by acting with creation operators α_n^μ and $\tilde{\alpha}_n^\mu$ with $n < 0$. A generic state comes from acting with any number of these creation operators on the vacuum,

$$(\alpha_{-1}^{\mu_1})^{n_{\mu_1}}\,(\alpha_{-2}^{\mu_2})^{n_{\mu_2}}\ldots(\tilde{\alpha}_{-1}^{\nu_1})^{n_{\nu_1}}\,(\tilde{\alpha}_{-2}^{\nu_2})^{n_{\nu_2}}\ldots|0;p\rangle$$

Each state in the Fock space is a different excited state of the string. Each has the interpretation of a different species of particle in spacetime. We'll see exactly what particles they are shortly. But for now, notice that because there's an infinite number of ways to excite a string there are an infinite number of different species of particles in this theory.

2.1.1 Ghosts

There's a problem with the Fock space that we've constructed: it doesn't have positive norm. The reason for this is that one of the scalar fields, X^0, comes with the wrong sign kinetic term in the action (1.30). From the perspective of the commutation relations, this issue raises its head in presence of the spacetime Minkowski metric in the expression

$$[\alpha_n^\mu, \alpha_m^{\nu\,\dagger}] = n\,\eta^{\mu\nu}\,\delta_{n,m}\ .$$

This gives rise to the offending negative norm states, which come with an odd number of timelike oscillators excited, for example

$$\langle p';0|\alpha_1^0\alpha_{-1}^0|0;p\rangle \sim -\delta^D(p-p')$$

This is the first problem that arises in the covariant approach to quantization. States with negative norm are referred to as *ghosts*. To make sense of the theory, we have to make sure that they can't be produced in any physical processes. Of course, this problem is familiar from attempts to quantize QED in Lorentz gauge. In that case, gauge symmetry rides to the rescue since the ghosts are removed by imposing the gauge fixing constraint. We must hope that the same happens in string theory.

2.1.2 Constraints

Although we won't push through with this programme at the present time, let us briefly look at what kind of constraints we have in string theory. In terms of Fourier modes, the classical constraints can be written as $L_n = \tilde{L}_n = 0$, where

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m$$

and similar for $\tilde{L}_n$. As in the Gupta-Bleuler quantization of QED, we don't impose all of these as operator equations on the Hilbert space. Instead we only require that the operators L_n and $\tilde{L}_n$ have vanishing matrix elements when sandwiched between two physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$,

$$\langle\text{phys}'|L_n|\text{phys}\rangle = \langle\text{phys}'|\tilde{L}_n|\text{phys}\rangle = 0$$

Because $L_n^\dagger = L_{-n}$, it is therefore sufficient to require

$$L_n|\text{phys}\rangle = \tilde{L}_n|\text{phys}\rangle = 0 \quad \text{for} \;\; n > 0 \tag{2.6}$$

However, we still haven't explained how to impose the constraints L_0 and $\tilde{L}_0$. And these present a problem that doesn't arise in the case of QED. The problem is that, unlike for L_n with $n \neq 0$, the operator L_0 is not uniquely defined when we pass to the quantum theory. There is an operator ordering ambiguity arising from the commutation relations (2.2). Commuting the α_n^μ operators past each other in L_0 gives rise to extra constant terms.

Question: How do we know what order to put the α_n^μ operators in the quantum operator L_0? Or the $\tilde{\alpha}_n^\mu$ operators in $\tilde{L}_0$?

Answer: We don't! Yet. Naively it looks as if each different choice will define a different theory when we impose the constraints. To make this ambiguity manifest, for now let's just pick a choice of ordering. We define the quantum operators to be normal ordered, with the annihilation operators α_n^i, $n > 0$, moved to the right,

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2}\,\alpha_0^2 \quad , \quad \tilde{L}_0 = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m + \frac{1}{2}\,\tilde{\alpha}_0^2$$

Then the ambiguity rears its head in the different constraint equations that we could impose, namely

$$(L_0 - a)|\text{phys}\rangle = (\tilde{L}_0 - a)|\text{phys}\rangle = 0 \tag{2.7}$$

for some constant a.

As we saw classically, the operators L_0 and $\tilde{L}_0$ play an important role in determining the spectrum of the string because they include a term quadratic in the momentum $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\,p^\mu$. Combining the expression (1.41) with our constraint equation for L_0 and $\tilde{L}_0$, we find the spectrum of the string is given by,

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m\right)$$

We learn therefore that the undetermined constant a has a direct physical effect: it changes the mass spectrum of the string. In the quantum theory, the sums over α_n^μ modes are related to the number operators for the harmonic oscillator: they count the number of excited modes of the string. The level matching in the quantum theory tells us that the number of left-moving modes must equal the number of right-moving modes.

Ultimately, we will find that the need to decouple the ghosts forces us to make a unique choice for the constant a. (Spoiler alert: it turns out to be $a = 1$). In fact, the requirement that there are no ghosts is much stronger than this. It also restricts the number of scalar fields that we have in the theory. (Another spoiler: $D = 26$). If you're interested in how this works in covariant formulation then you can read about it in the book by Green, Schwarz and Witten. Instead, we'll show how to quantize the string and derive these values for a and D in lightcone gauge. However, after a trip through the world of conformal field theory, we'll come back to these ideas in a context which is closer to the covariant approach.

2.2 Lightcone Quantization

We will now take the second path described at the beginning of this section. We will try to find a parameterization of all classical solutions of the string. This is equivalent to finding the classical phase space of the theory. We do this by solving the constraints (2.1) in the classical theory, leaving behind only the physical degrees of freedom.

Recall that we fixed the gauge to set the worldsheet metric to

$$g_{\alpha\beta} = \eta_{\alpha\beta} \ .$$

However, this isn't the end of our gauge freedom. There still remain gauge transformations which preserve this choice of metric. In particular, any coordinate transformation $\sigma \to \tilde{\sigma}(\sigma)$ which changes the metric by

$$\eta_{\alpha\beta} \to \Omega^2(\sigma)\eta_{\alpha\beta} \ , \tag{2.8}$$

can be undone by a Weyl transformation. What are these coordinate transformations? It's simplest to answer this using lightcone coordinates on the worldsheet,

$$\sigma^\pm = \tau \pm \sigma \ , \tag{2.9}$$

where the flat metric on the worldsheet takes the form,

$$ds^2 = -d\sigma^+ d\sigma^-$$

In these coordinates, it's clear that any transformation of the form

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad , \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \ , \tag{2.10}$$

simply multiplies the flat metric by an overall factor (2.8) and so can be undone by a compensating Weyl transformation. Some quick comments on this surviving gauge symmetry:

- Recall that in Section 1.3.2 we used the argument that 3 gauge invariances (2 reparameterizations + 1 Weyl) could be used to fix 3 components of the worldsheet metric $g_{\alpha\beta}$. What happened to this argument? Why do we still have some gauge symmetry left? The reason is that $\tilde{\sigma}^{\pm}$ are functions of just a single variable, not two. So we did fix nearly all our gauge symmetries. What is left is a set of measure zero amongst the full gauge symmetry that we started with.

- The remaining reparameterization invariance (2.10) has an important physical implication. Recall that the solutions to the equations of motion are of the form $X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ which looks like $2D$ functions worth of solutions. Of course, we still have the constraints which, in terms of $\sigma^\pm$, read

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \ , \tag{2.11}$$

  which seems to bring the number down to $2(D-1)$ functions. But the reparameterization invariance (2.10) tells us that even some of these are fake since we can always change what we mean by $\sigma^\pm$. The physical solutions of the string are therefore actually described by $2(D-2)$ functions. But this counting has a nice interpretation: the degrees of freedom describe the *transverse* fluctuations of the string.

- The above comment reaches the same conclusion as the discussion in Section 1.3.2. There, in an attempt to get some feel for the constraints, we claimed that we could go to static gauge $X^0 = R\tau$ for some dimensionful parameter R. It is easy to check that this is simple to do using reparameterizations of the form (2.10). However, to solve the string constraints in full, it turns out that static gauge is not that useful. Rather we will use something called "lightcone gauge".

2.2.1 Lightcone Gauge

We would like to gauge fix the remaining reparameterization invariance (2.10). The best way to do this is called lightcone gauge. In counterpoint to the worldsheet lightcone coordinates (2.9), we introduce the spacetime lightcone coordinates,

$$X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}) \ . \tag{2.12}$$

Note that this choice picks out a particular time direction and a particular spatial direction. It means that any calculations that we do involving $X^\pm$ will not be manifestly Lorentz invariant. You might think that we needn't really worry about this. We could try to make the following argument: "The equations may not *look* Lorentz invariant but, since we started from a Lorentz invariant theory, at the end of the day any physical process is guaranteed to obey this symmetry". Right?! Well, unfortunately not. One of the more interesting and subtle aspects of quantum field theory is the possibility of anomalies: these are symmetries of the classical theory that do not survive the journey of quantization. When we come to the quantum theory, if our equations don't look Lorentz invariant then there's a real possibility that it's because the underlying physics actually isn't Lorentz invariant. Later we will need to spend some time figuring out under what circumstances our quantum theory keeps the classical Lorentz symmetry.

In lightcone coordinates, the spacetime Minkowski metric reads

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i$$

This means that indices are raised and lowered with $A_+ = -A^-$ and $A_- = -A^+$ and $A_i = A^i$. The product of spacetime vectors reads $A \cdot B = -A^+B^- - A^-B^+ + A^iB^i$.

Let's look at the solution to the equation of motion for X^+. It reads,

$$X^+ = X^+_L(\sigma^+) + X^+_R(\sigma^-)\ .$$

We now gauge fix. We use our freedom of reparameterization invariance to choose coordinates such that

$$X^+_L = \tfrac{1}{2}\,x^+ + \tfrac{1}{2}\,\alpha' p^+ \sigma^+ \quad , \quad X^+_R = \tfrac{1}{2}\,x^+ + \tfrac{1}{2}\,\alpha' p^+ \sigma^-\ .$$

You might think that we could go further and eliminate p^+ and x^+ but this isn't possible because we don't quite have the full freedom of reparameterization invariance since all functions should remain periodic in σ. The upshot of this choice of gauge is that

$$X^+ = x^+ + \alpha' p^+ \tau\ . \tag{2.13}$$

This is *lightcone gauge*. Notice that, as long as $p^+ \neq 0$, we can always shift x^+ by a shift in τ.

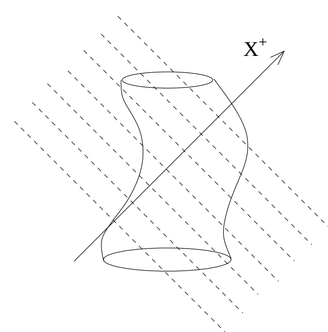

Figure 9:

There's something a little disconcerting about the choice (2.13). We've identified a timelike worldsheet coordinate with a null spacetime coordinate. Nonetheless, as you can see from the figure, it seems to be a good parameterization of the worldsheet. One could imagine that the parameterization might break if the string is actually massless and travels in the X^- direction, with $p^+ = 0$. But otherwise, all should be fine.

Solving for X^-

The choice (2.13) does the job of fixing the reparameterization invariance (2.10). As we will now see, it also renders the constraint equations trivial. The first thing that we have to worry about is the possibility of extra constraints arising from this new choice of gauge fixing. This can be checked by looking at the equation of motion for X^+,

$$\partial_+\partial_- X^- = 0$$

But we can solve this by the usual ansatz,

$$X^- = X^-_L(\sigma^+) + X^-_R(\sigma^-) \ .$$

We're still left with all the other constraints (2.11). Here we see the real benefit of working in lightcone gauge (which is actually what makes quantization possible at all): X^- is completely determined by these constraints. For example, the first of these reads

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{2.14}$$

which, using (2.13), simply becomes

$$\partial_+ X^-_L = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \ . \tag{2.15}$$

Similarly,

$$\partial_- X^-_R = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i \ . \tag{2.16}$$

So, up to an integration constant, the function $X^-(\sigma^+, \sigma^-)$ is completely determined in terms of the other fields. If we write the usual mode expansion for $X^-_{L/R}$

$$X^-_L(\sigma^+) = \tfrac{1}{2}x^- + \tfrac{1}{2}\alpha' p^- \, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}^-_n \, e^{-in\sigma^+} \ ,$$
$$X^-_R(\sigma^-) = \tfrac{1}{2}x^- + \tfrac{1}{2}\alpha' p^- \, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \alpha^-_n \, e^{-in\sigma^-} \ .$$

then x^- is the undetermined integration constant, while p^-, α^-_n and $\tilde{\alpha}^-_n$ are all fixed by the constraints (2.15) and (2.16). For example, the oscillator modes α^-_n are given by,

$$\alpha^-_n = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha^i_{n-m} \alpha^i_m \ , \tag{2.17}$$

A special case of this is the $\alpha_0^- = \sqrt{\alpha'/2}\, p^-$ equation, which reads

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\tfrac{1}{2}\,\alpha' p^i p^i + \sum_{n\neq 0}\alpha_n^i \alpha_{-n}^i\right)\ . \tag{2.18}$$

We also get another equation for p^- from the $\tilde{\alpha}_0^-$ equation arising from (2.15)

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\tfrac{1}{2}\,\alpha' p^i p^i + \sum_{n\neq 0}\tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right)\ . \tag{2.19}$$

From these two equations, we can reconstruct the old, classical, level matching conditions (1.41). But now with a difference:

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\ . \tag{2.20}$$

The difference is that now the sum is over oscillators α^i and $\tilde{\alpha}^i$ only, with $i = 1, \ldots, D-2$. We'll refer to these as *transverse* oscillators. Note that the string isn't necessarily living in the X^0-X^{D-1} plane, so these aren't literally the transverse excitations of the string. Nonetheless, if we specify the α^i then all other oscillator modes are determined. In this sense, they are the physical excitation of the string.

Let's summarize the state of play so far. The most general classical solution is described in terms of $2(D-2)$ transverse oscillator modes α_n^i and $\tilde{\alpha}_n^i$, together with a number of zero modes describing the center of mass and momentum of the string: x^i, p^i, p^+ and x^-. But x^+ can be absorbed by a shift of τ in (2.13) and p^- is constrained to obey (2.18) and (2.19). In fact, p^- can be thought of as (proportional to) the lightcone Hamiltonian. Indeed, we know that p^- generates translations in x^+, but this is equivalent to shifts in τ.

2.2.2 Quantization

Having identified the physical degrees of freedom, let's now quantize. We want to impose commutation relations. Some of these are easy:

$$\begin{gathered}[x^i, p^j] = i\delta^{ij} \quad , \quad [x^-, p^+] = -i \\ [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}\ .\end{gathered} \tag{2.21}$$

all of which follow from the commutation relations (2.2) that we saw in covariant quantization[1].

What to do with x^+ and p^-? We could implement p^- as the Hamiltonian acting on states. In fact, it will prove slightly more elegant (but equivalent) if we promote both x^+ and p^- to operators with the expected commutation relation,

$$[x^+, p^-] = -i \ . \tag{2.22}$$

This is morally equivalent to writing $[t, H] = -i$ in non-relativistic quantum mechanics, which is true on a formal level. In the present context, it means that we can once again choose states to be eigenstates of p^μ, with $\mu = 0, \ldots, D$, but the constraints (2.18) and (2.19) must still be imposed as operator equations on the physical states. We'll come to this shortly.

The Hilbert space of states is very similar to that described in covariant quantization: we define a vacuum state, $|0;p\rangle$ such that

$$\hat{p}^\mu |0;p\rangle = p^\mu |0;p^\mu\rangle \quad , \quad \alpha_n^i |0;p\rangle = \tilde{\alpha}_n^i |0;p\rangle = 0 \quad \text{for} \ \ n > 0 \tag{2.23}$$

and we build a Fock space by acting with the creation operators α_{-n}^i and $\tilde{\alpha}_{-n}^i$ with $n > 0$. The difference with the covariant quantization is that we only act with transverse oscillators which carry a spatial index $i = 1, \ldots, D-2$. For this reason, the Hilbert space is, by construction, positive definite. We don't have to worry about ghosts.

[1]**Mea Culpa:** We're not really supposed to do this. The whole point of the approach that we're taking is to quantize just the physical degrees of freedom. The resulting commutation relations are not, in general, inherited from the larger theory that we started with simply by closing our eyes and forgetting about all the other fields that we've gauge fixed. We can see the problem by looking at (2.17), where α_n^- is determined in terms of α_n^i. This means that the commutation relations for α_n^i might be infected by those of α_n^- which could potentially give rise to extra terms. The correct procedure to deal with this is to figure out the Poisson bracket structure of the physical degrees of freedom in the classical theory. Or, in fancier language, the symplectic form on the phase space which schematically looks like

$$\omega \sim \int d\sigma \ \ - d\dot{X}^+ \wedge dX^- - d\dot{X}^- \wedge dX^+ + 2d\dot{X}^i \wedge dX^i \ ,$$

The reason that the commutation relations (2.21) do not get infected is because the α^- terms in the symplectic form come multiplying X^+. Yet X^+ is given in (2.13). It has no oscillator modes. That means that the symplectic form doesn't pick up the Fourier modes of X^-, and so doesn't receive any corrections from α_n^-. The upshot of this is that the naive commutation relations (2.21) are actually right.

The Constraints

Because p^- is not an independent variable in our theory, we must impose the constraints (2.18) and (2.19) by hand as operator equations which define the physical states. In the classical theory, we saw that these constraints are equivalent to mass-shell conditions (2.20).

But there's a problem when we go to the quantum theory. It's the same problem that we saw in covariant quantization: there's an ordering ambiguity in the sum over oscillator modes on the right-hand side of (2.20). If we choose all operators to be normal ordered then this ambiguity reveals itself in an overall constant, a, which we have not yet determined. The final result for the mass of states in lightcone gauge is:

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - a\right)$$

Since we'll use this formula quite a lot in what follows, it's useful to introduce quantities related to the number operators of the harmonic oscillator,

$$N = \sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \quad , \quad \tilde{N} = \sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \ . \tag{2.24}$$

These are not quite number operators because of the factor of $1/\sqrt{n}$ in (2.3). The value of N and $\tilde{N}$ is often called the level. Which, if nothing else, means that the name "level matching" makes sense. We now have

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a) \ . \tag{2.25}$$

How are we going to fix a? Later in the course we'll see the correct way to do it. For now, I'm just going to give you a quick and dirty derivation.

The Casimir Energy

What follows is a heuristic derivation of the normal ordering constant a. Suppose that we didn't notice that there was any ordering ambiguity and instead took the naive classical result directly over to the quantum theory, that is

$$\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{n<0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \ .$$

where we've left the sum over $i = 1, \ldots, D-2$ implicit. We'll now try to put this in normal ordered form, with the annihilation operators α^i_n with $n > 0$ on the right-hand side. It's the first term that needs changing. We get

$$\frac{1}{2}\sum_{n<0}\left[\alpha_n^i\alpha_{-n}^i - n(D-2)\right] + \frac{1}{2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i = \sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0} n \quad .$$

The final term clearly diverges. But it at least seems to have a physical interpretation: it is the sum of zero point energies of an infinite number of harmonic oscillators. In fact, we came across exactly the same type of term in the course on quantum field theory where we learnt that, despite the divergence, one can still extract interesting physics from this. This is the physics of the Casimir force.

Let's recall the steps that we took to derive the Casimir force. Firstly, we introduced an ultra-violet cut-off $\epsilon \ll 1$, probably muttering some words about no physical plates being able to withstand very high energy quanta. Unfortunately, those words are no longer available to us in string theory, but let's proceed regardless. We replace the divergent sum over integers by the expression,

$$\begin{aligned}\sum_{n=1}^{\infty} n \;\longrightarrow\; \sum_{n=1}^{\infty} n e^{-\epsilon n} &= -\frac{\partial}{\partial\epsilon}\sum_{n=1}^{\infty} e^{-\epsilon n} \\ &= -\frac{\partial}{\partial\epsilon}(1-e^{-\epsilon})^{-1} \\ &= \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)\end{aligned}$$

Obviously the $1/\epsilon$ piece diverges as $\epsilon \to 0$. This term should be renormalized away. In fact, this is necessary to preserve the Weyl invariance of the Polyakov action since it contributes to a cosmological constant on the worldsheet. After this renormalization, we're left with the answer

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \; .$$

While heuristic, this argument does predict the correct physical Casimir energy measured in one-dimensional systems. For example, this effect is seen in simulations of quantum spin chains.

What does this mean for our string? It means that we should take the unknown constant a in the mass formula (2.25) to be,

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right) \quad . \tag{2.26}$$

This is the formula that we will use to determine the spectrum of the string.

Zeta Function Regularization

I appreciate that the preceding argument is not totally convincing. We could spend some time making it more robust at this stage, but it's best if we wait until later in the course when we will have the tools of conformal field theory at our disposal. We will eventually revisit this issue and provide a respectable derivation of the Casimir energy in Section 4.4.1. For now I merely offer an even less convincing argument, known as zeta-function regularization.

The zeta-function is defined, for $\text{Re}(s) > 1$, by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ .$$

But $\zeta(s)$ has a unique analytic continuation to all values of s. In particular,

$$\zeta(-1) = -\frac{1}{12} \ .$$

Good? Good. This argument is famously unconvincing the first time you meet it! But it's actually a very useful trick for getting the right answer.

2.3 The String Spectrum

Finally, we're in a position to analyze the spectrum of a single, free string.

2.3.1 The Tachyon

Let's start with the ground state $|0;p\rangle$ defined in (2.23). With no oscillators excited, the mass formula (2.26) gives

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6} \ . \tag{2.27}$$

But that's a little odd. It's a negative mass-squared. Such particles are called *tachyons*.

In fact, tachyons aren't quite as pathological as you might think. If you've heard of these objects before, it's probably in the context of special relativity where they're strange beasts which always travel faster than the speed of light. But that's not the right interpretation. Rather we should think more in the language of quantum field theory. Suppose that we have a field in spacetime — let's call it $T(X)$ — whose quanta will give rise to this particle. The mass-squared of the particle is simply the quadratic term in the action, or

$$M^2 = \left.\frac{\partial^2 V(T)}{\partial T^2}\right|_{T=0}$$

So the negative mass-squared in (2.27) is telling us that we're expanding around a maximum of the potential for the tachyon field as shown in the figure. Note that from this perspective, the Higgs field in the standard model at $H = 0$ is also a tachyon.

The fact that string theory turns out to sit at an unstable point in the tachyon field is unfortunate. The natural question is whether the potential has a good minimum elsewhere, as shown in the figure to the right. No one knows the answer to this! Naive attempts to understand this don't work. We know that around $T = 0$, the leading order contribution to the potential is negative and quadratic. But there are further terms that we can compute using techniques that we'll describe in Section 6. An expansion of the tachyon potential around $T = 0$ looks like

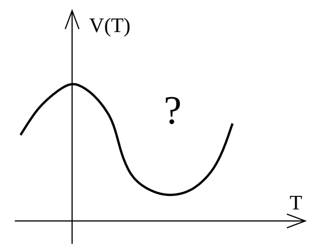

Figure 11:

$$V(T) = \frac{1}{2}M^2T^2 + c_3T^3 + c_4T^4 + \ldots$$

It turns out that the T^3 term in the potential does give rise to a minimum. But the T^4 term destabilizes it again. Moreover, the T field starts to mix with other scalar fields in the theory that we will come across soon. The ultimate fate of the tachyon in the bosonic string is not yet understood.

The tachyon is a problem for the bosonic string. It may well be that this theory makes no sense — or, at the very least, has no time-independent stable solutions. Or perhaps we just haven't worked out how to correctly deal with the tachyon. Either way, the problem does not arise when we introduce fermions on the worldsheet and study the superstring. This will involve several further technicalities which we won't get into in this course. Instead, our time will be put to better use if we continue to study the bosonic string since all the lessons that we learn will carry over directly to the superstring. However, one should be aware that the problem of the unstable vacuum will continue to haunt us throughout this course.

Although we won't describe it in detail, at several times along our journey we'll make an aside about how calculations work out for the superstring.

2.3.2 The First Excited States

We now look at the first excited states. If we act with a creation operator α^j_{-1}, then the level matching condition (2.25) tells us that we also need to act with a $\tilde{\alpha}^i_{-1}$ operator. This gives us $(D-2)^2$ particle states,

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle \;, \tag{2.28}$$

each of which has mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) \ .$$

But now we seem to have a problem. Our states have space indices $i, j = 1, \ldots D-2$. The operators α^i and $\tilde{\alpha}^i$ each transform in the vector representation of $SO(D-2) \subset SO(1, D-1)$ which is manifest in lightcone gauge. But ultimately we want these states to fit into some representation of the full Lorentz $SO(1, D-1)$ group. That looks as if it's going to be hard to arrange. This is the first manifestation of the comment that we made after equation (2.12): it's tricky to see Lorentz invariance in lightcone gauge.

To proceed, let's recall Wigner's classification of representations of the Poincaré group. We start by looking at massive particles in $\mathbf{R}^{1,D-1}$. After going to the rest frame of the particle by setting $p^\mu = (p, 0 \ldots, 0)$, we can watch how any internal indices transform under the little group $SO(D-1)$ of spatial rotations. The upshot of this is that any massive particle must form a representation of $SO(D-1)$. But the particles described by (2.28) have $(D-2)^2$ states. There's no way to package these states into a representation of $SO(D-1)$ and this means that there's no way that the first excited states of the string can form a massive representation of the D-dimensional Poincaré group.

It looks like we're in trouble. Thankfully, there's a way out. If the states are massless, then we can't go to the rest frame. The best that we can do is choose a spacetime momentum for the particle of the form $p^\mu = (p, 0, \ldots, 0, p)$. In this case, the particles fill out a representation of the little group $SO(D-2)$. This means that massless particles get away with having fewer internal states than massive particles. For example, in four dimensions the photon has two polarization states, but a massive spin-1 particle must have three.

The first excited states (2.28) happily sit in a representation of $SO(D-2)$. We learn that if we want the quantum theory to preserve the $SO(1, D-1)$ Lorentz symmetry that we started with, then these states will have to be massless. And this is only the case if the dimension of spacetime is

$$D = 26 \ .$$

This is our first derivation of the critical dimension of the bosonic string.

Moreover, we've found that our theory contains a bunch of massless particles. And massless particles are interesting because they give rise to long range forces. Let's look more closely at what massless particles the string has given us. The states (2.28) transform in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$. These decompose into three irreducible representations:

$$\text{traceless symmetric} \;\oplus\; \text{anti-symmetric} \;\oplus\; \text{singlet (=trace)}$$

To each of these modes, we associate a massless field in spacetime such that the string oscillation can be identified with a quantum of these fields. The fields are:

$$G_{\mu\nu}(X) \quad , \quad B_{\mu\nu}(X) \quad , \quad \Phi(X) \tag{2.29}$$

Of these, the first is the most interesting and we shall have more to say momentarily. The second is an anti-symmetric tensor field which is usually called the anti-symmetric tensor field. It also goes by the names of the "Kalb-Ramond field" or, in the language of differential geometry, the "2-form". The scalar field is called the *dilaton*. These three massless fields are common to all string theories. We'll learn more about the role these fields play later in the course.

The particle in the symmetric traceless representation of $SO(24)$ is particularly interesting. This is a massless spin 2 particle. However, there are general arguments, due originally to Feynman and Weinberg, that *any* theory of interacting massless spin two particles must be equivalent to general relativity[2]. We should therefore identify the field $G_{\mu\nu}(X)$ with the metric of spacetime. Let's pause briefly to review the thrust of these arguments.

Why Massless Spin 2 = General Relativity

Let's call the spacetime metric $G_{\mu\nu}(X)$. We can expand around flat space by writing

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(X) \ .$$

Then the Einstein-Hilbert action has an expansion in powers of h. If we truncate to quadratic order, we simply have a free theory which we may merrily quantize in the usual canonical fashion: we promote $h_{\mu\nu}$ to an operator and introduce the associated creation and annihilation operators $a_{\mu\nu}$ and $a^\dagger_{\mu\nu}$. This way of looking at gravity is anathema to those raised in the geometrical world of general relativity. But from a particle physics language it is very standard: it is simply the quantization of a massless spin 2 field, $h_{\mu\nu}$.

[2] A very readable description of this can be found in the first few chapters of the Feynman Lectures on Gravitation.

However, even on this simple level, there is a problem due to the indefinite signature of the spacetime Minkowski metric. The canonical quantization relations of the creation and annihilation operators are schematically of the form,

$$[a_{\mu\nu}, a^\dagger_{\rho\sigma}] \sim \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}$$

But this will lead to a Hilbert space with negative norm states coming from acting with time-like creation operators. For example, the one-graviton state of the form,

$$a^\dagger_{0i}|0\rangle \tag{2.30}$$

suffers from a negative norm. This should be becoming familiar by now: it is the usual problem that we run into if we try to covariantly quantize a gauge theory. And, indeed, general relativity is a gauge theory. The gauge transformations are diffeomorphisms. We would hope that this saves the theory of quantum gravity from these negative norm states.

Let's look a little more closely at what the gauge symmetry looks like for small fluctuations $h_{\mu\nu}$. We've butchered the Einstein-Hilbert action and left only terms quadratic in h. Including all the index contractions, we find

$$S_{EH} = \frac{M_{pl}^2}{2}\int d^4x\ \left[\partial_\mu h^\rho{}_\rho \partial_\nu h^{\mu\nu} - \partial^\rho h^{\mu\nu}\partial_\mu h_{\rho\nu} + \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{2}\partial_\mu h^\nu{}_\nu \partial^\mu h^\rho{}_\rho\right] + \ldots$$

One can check that this truncated action is invariant under the gauge symmetry,

$$h_{\mu\nu} \ \longrightarrow\ h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{2.31}$$

for any function $\xi_\mu(X)$. The gauge symmetry is the remnant of diffeomorphism invariance, restricted to small deviations away from flat space. With this gauge invariance in hand one can show that, just like QED, the negative norm states decouple from all physical processes.

To summarize, theories of massless spin 2 fields only make sense if there is a gauge symmetry to remove the negative norm states. In general relativity, this gauge symmetry descends from diffeomorphism invariance. The argument of Feynman and Weinberg now runs this logic in reverse. It goes as follows: suppose that we have a massless, spin 2 particle. Then, at the linearized level, it must be invariant under the gauge symmetry (2.31) in order to eliminate the negative norm states. Moreover, this symmetry must survive when interaction terms are introduced. But the only way to do this is to ensure that the resulting theory obeys diffeomorpism invariance. That means the theory of any interacting, massless spin 2 particle is Einstein gravity, perhaps supplemented by higher derivative terms.

We haven't yet shown that string theory includes interactions for $h_{\mu\nu}$ but we will come to this later in the course. More importantly, we will also explicitly see how Einstein's field equations arise directly in string theory.

A Comment on Spacetime Gauge Invariance

We've surreptitiously put $\mu, \nu = 0, \ldots, 25$ indices on the spacetime fields, rather than $i, j = 1, \ldots, 24$. The reason we're allowed to do this is because both $G_{\mu\nu}$ and $B_{\mu\nu}$ enjoy a spacetime gauge symmetry which allows us to eliminate appropriate modes. Indeed, this is exactly the gauge symmetry (2.31) that entered the discussion above. It isn't possible to see these spacetime gauge symmetries from the lightcone formalism of the string since, by construction, we find only the physical states (although, by consistency alone, the gauge symmetries must be there). One of the main advantages of pushing through with the covariant calculation is that it does allow us to see how the spacetime gauge symmetry emerges from the string worldsheet. Details can be found in Green, Schwarz and Witten. We'll also briefly return to this issue in Section 5.

2.3.3 Higher Excited States

We rescued the Lorentz invariance of the first excited states by choosing $D = 26$ to ensure that they are massless. But now we've used this trick once, we still have to worry about all the other excited states. These also carry indices that take the range $i, j = 1, \ldots, D-2 = 24$ and, from the mass formula (2.26), they will all be massive and so must form representations of $SO(D-1)$. It looks like we're in trouble again.

Let's examine the string at level $N = \tilde{N} = 2$. In the right-moving sector, we now have two different states: $\alpha^i_{-1}\alpha^j_{-1}|0\rangle$ and $\alpha^i_{-2}|0\rangle$. The same is true for the left-moving sector, meaning that the total set of states at level 2 is (in notation that is hopefully obvious, but probably technically wrong)

$$(\alpha^i_{-1}\alpha^j_{-1} \oplus \alpha^i_{-2}) \otimes (\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1} \oplus \tilde{\alpha}^i_{-2})|0; p\rangle \ .$$

These states have mass $M^2 = 4/\alpha'$. How many states do we have? In the left-moving sector, we have,

$$\tfrac{1}{2}(D-2)(D-1) + (D-2) = \tfrac{1}{2}D(D-1) - 1 \ .$$

But, remarkably, that does fit nicely into a representation of $SO(D-1)$, namely the traceless symmetric tensor representation.

In fact, one can show that all excited states of the string fit nicely into $SO(D-1)$ representations. The only consistency requirement that we need for Lorentz invariance is to fix up the first excited states: $D = 26$.

Note that if we are interested in a fundamental theory of quantum gravity, then all these excited states will have masses close to the Planck scale so are unlikely to be observable in particle physics experiments. Nonetheless, as we shall see when we come to discuss scattering amplitudes, it is the presence of this infinite tower of states that tames the ultra-violet behaviour of gravity.

2.4 Lorentz Invariance Revisited

The previous discussion allowed to us to derive both the critical dimension and the spectrum of string theory in the quickest fashion. But the derivation creaks a little in places. The calculation of the Casimir energy is unsatisfactory the first time one sees it. Similarly, the explanation of the need for massless particles at the first excited level is correct, but seems rather cheap considering the huge importance that we're placing on the result.

As I've mentioned a few times already, we'll shortly do better and gain some physical insight into these issues, in particular the critical dimension. But here I would just like to briefly sketch how one can be a little more rigorous within the framework of lightcone quantization. The question, as we've seen, is whether one preserves spacetime Lorentz symmetry when we quantize in lightcone gauge. We can examine this more closely.

Firstly, let's go back to the action for free scalar fields (1.30) before we imposed lightcone gauge fixing. Here the full Poincaré symmetry was manifest: it appears as a global symmetry on the worldsheet,

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{2.32}$$

But recall that in field theory, global symmetries give rise to Noether currents and their associated conserved charges. What are the Noether currents associated to this Poincaré transformation? We can start with the translations $X^\mu \to X^\mu + c^\mu$. A quick computation shows that the current is,

$$P^\alpha_\mu = T\partial^\alpha X_\mu \tag{2.33}$$

which is indeed a conserved current since $\partial_\alpha P^\alpha_\mu = 0$ is simply the equation of motion. Similarly, we can compute the $\frac{1}{2}D(D-1)$ currents associated to Lorentz transformations. They are,

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu$$

It's not hard to check that $\partial_\alpha J^\alpha_{\mu\nu} = 0$ when the equations of motion are obeyed.

The conserved charges arising from this current are given by $M_{\mu\nu} = \int d\sigma J^\tau_{\mu\nu}$. Using the mode expansion (1.36) for X^μ, these can be written as

$$\begin{aligned}\mathcal{M}^{\mu\nu} &= (p^\mu x^\nu - p^\nu x^\mu) - i\sum_{n=1}^\infty \frac{1}{n}\left(\alpha^\nu_{-n}\alpha^\mu_n - \alpha^\mu_{-n}\alpha^\nu_n\right) - i\sum_{n=1}^\infty \frac{1}{n}\left(\tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n - \tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n\right)\\ &\equiv l^{\mu\nu} + S^{\mu\nu} + \tilde{S}^{\mu\nu}\end{aligned}$$

The first piece, $l^{\mu\nu}$, is the orbital angular momentum of the string while the remaining pieces $S^{\mu\nu}$ and $\tilde{S}^{\mu\nu}$ tell us the angular momentum due to excited oscillator modes. Classically, these obey the Poisson brackets of the Lorentz algebra. Moreover, if we quantize in the covariant approach, the corresponding operators obey the commutation relations of the Lorentz Lie algebra, namely

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}$$

However, things aren't so easy in lightcone gauge. Lorentz invariance is not guaranteed and, in general, is not there. The right way to go about looking for it is to make sure that the Lorentz algebra above is reproduced by the generators $\mathcal{M}^{\mu\nu}$. It turns out that the smoking gun lies in the commutation relation,

$$[\mathcal{M}^{i-}, \mathcal{M}^{j-}] = 0$$

Does this equation hold in lightcone gauge? The problem is that it involves the operators p^- and α^-_n, both of which are fixed by (2.17) and (2.18) in terms of the other operators. So the task is to compute this commutation relation $[\mathcal{M}^{i-}, \mathcal{M}^{j-}]$, given the commutation relations (2.21) for the physical degrees of freedom, and check that it vanishes. To do this, we re-instate the ordering ambiguity a and the number of spacetime dimension D as arbitrary variables and proceed.

The part involving orbital angular momenta l^{i-} is fairly straightforward. (Actually, there's a small subtlety because we must first make sure that the operator $l^{\mu\nu}$ is Hermitian by replacing $x^\mu p^\nu$ with $\frac{1}{2}(x^\mu p^\nu + p^\nu x^\mu)$). The real difficulty comes from computing the commutation relations $[S^{i-}, S^{j-}]$. This is messy[3]. After a tedious computation, one finds,

$$[\mathcal{M}^{i-}, \mathcal{M}^{j-}] = \frac{2}{(p^+)^2}\sum_{n>0}\left(\left[\frac{D-2}{24} - 1\right]n + \frac{1}{n}\left[a - \frac{D-2}{24}\right]\right)(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) + (\alpha \leftrightarrow \tilde{\alpha})$$

[3]The original, classic, paper where lightcone quantization was first implemented is Goddard, Goldstone, Rebbi and Thorn "*Quantum Dynamics of a Massless Relativistic String*", Nucl. Phys. B56 (1973). A pedestrian walkthrough of this calculation can be found in the lecture notes by Gleb Arutyunov. A link is given on the course webpage.

The right-hand side does not, in general, vanish. We learn that the relativistic string can only be quantized in flat Minkowski space if we pick,

$$D = 26 \quad \text{and} \quad a = 1 \ .$$

2.5 A Nod to the Superstring

We won't provide details of the superstring in this course, but will pause occasionally to make some pertinent comments. Although what follows is nothing more than a list of facts, it will hopefully be helpful in orienting you when you do come to study this material.

The key difference between the bosonic string and the superstring is the addition of fermionic modes on its worldsheet. The resulting worldsheet theory is supersymmetric. (At least in the so-called Neveu-Schwarz-Ramond formalism). Hence the name "superstring". Applying the kind of quantization procedure we've discussed in this section, one finds the following results:

- The critical dimension of the superstring is $D = 10$.
- There is no tachyon in the spectrum.
- The massless bosonic fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are all part of the spectrum of the superstring. In this context, $B_{\mu\nu}$ is sometimes referred to as the Neveu-Schwarz 2-form. There are also massless spacetime fermions, as well as further massless bosonic fields. As we now discuss, the exact form of these extra bosonic fields depends on exactly what superstring theory we consider.

While the bosonic string is unique, there are a number of discrete choices that one can make when adding fermions to the worldsheet. This gives rise to a handful of different perturbative superstring theories. (Although later developments reveal that they are actually all part of the same framework which sometimes goes by the name of *M-theory*). The most important of these discrete options is whether we add fermions in both the left-moving and right-moving sectors of the string, or whether we choose the fermions to move only in one direction, usually taken to be right-moving. This gives rise to two different classes of string theory.

- Type II strings have both left and right-moving worldsheet fermions. The resulting spacetime theory in $D = 10$ dimensions has $\mathcal{N} = 2$ supersymmetry, which means 32 supercharges.
- Heterotic strings have just right-moving fermions. The resulting spacetime theory has $\mathcal{N} = 1$ supersymmetry, or 16 supercharges.

In each of these cases, there is then one further discrete choice that we can make. This leaves us with four superstring theories. In each case, the massless bosonic fields include $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ together with a number of extra fields. These are:

- **Type IIA**: In the type II theories, the extra massless bosonic excitations of the string are referred to as *Ramond-Ramond* fields. For Type IIA, they are a 1-form C_μ and a 3-form $C_{\mu\nu\rho}$. Each of these is to be thought of as a gauge field. The gauge invariant information lies in the field strengths which take the form $F = dC$.

- **Type IIB**: The Ramond-Ramond gauge fields consist of a scalar C, a 2-form $C_{\mu\nu}$ and a 4-form $C_{\mu\nu\rho\sigma}$. The 4-form is restricted to have a self-dual field strength: $F_5 = {}^\star F_5$. (Actually, this statement is almost true...we'll look a little closer at this in Section 7.3.3).

- **Heterotic** $SO(32)$: The heterotic strings do not have Ramond-Ramond fields. Instead, each comes with a non-Abelian gauge field in spacetime. The heterotic strings are named after the gauge group. For example, the Heterotic $SO(32)$ string gives rise to an $SO(32)$ Yang-Mills theory in ten dimensions.

- **Heterotic** $E_8 \times E_8$: The clue is in the name. This string gives rise to an $E_8 \times E_8$ Yang-Mills field in ten-dimensions.

It is sometimes said that there are five perturbative superstring theories in ten dimensions. Here we've only mentioned four. The remaining theory is called Type I and includes open strings moving in flat ten dimensional space as well as closed strings. We'll mention it in passing in the following section.